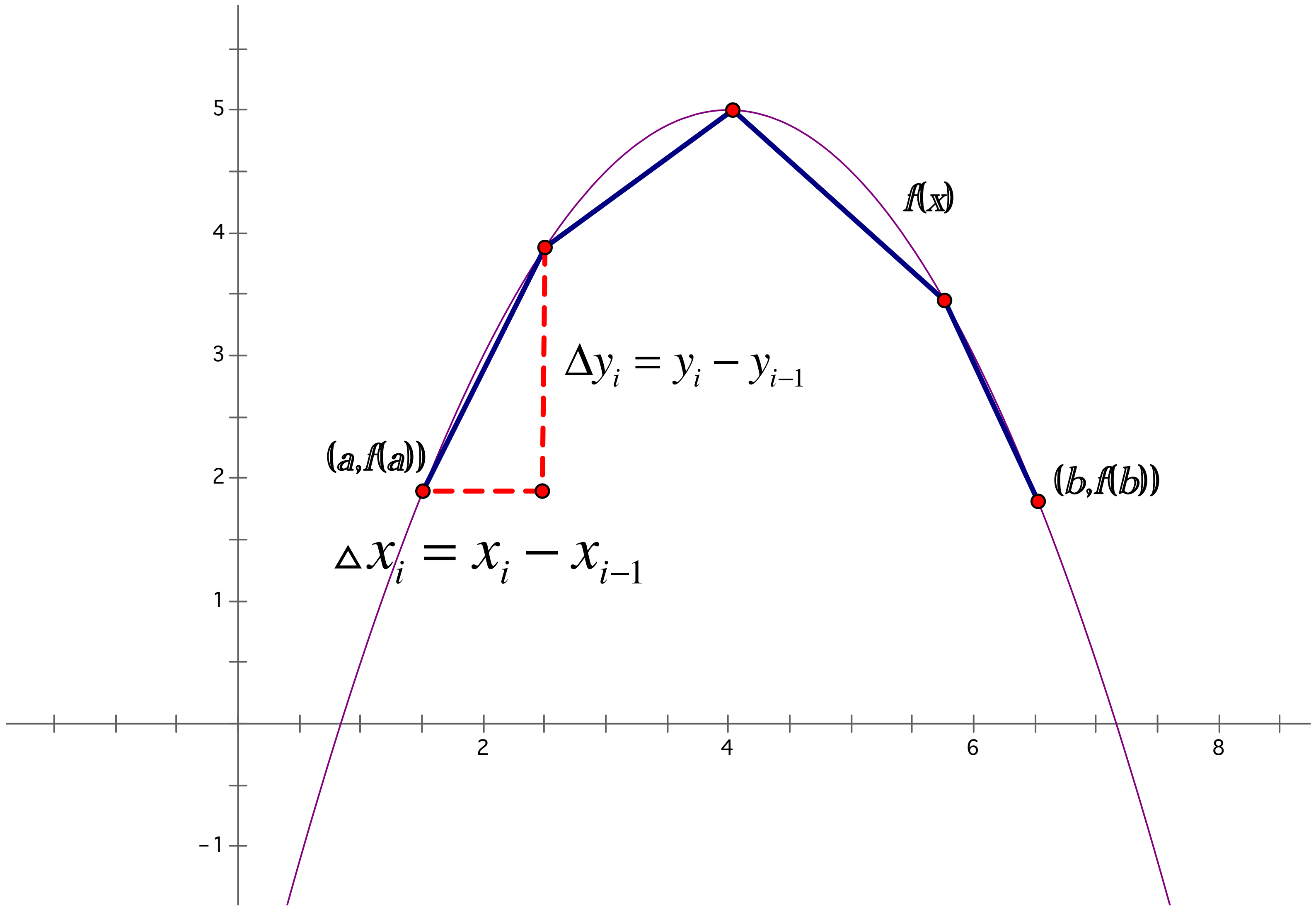
Arc Length & Surfaces of Revolution (6.4)

January 31st, 2019

Arc Length

We can calculate an arc length of a curve by integrating an expression that uses the distance of a series of straight line segments going from one endpoint of the arc to the other. Integrating this expression gives us the limit as the number of line segments being used approaches ∞.

Proof of arc length formula:



*The slope of the line segment and the derivative of f can be substituted as a result of the Mean Value Theorem.

Definition of Arc Length

Let f be a function given by y=f(x) represent a smooth curve on the interval [a, b]. The <u>arc length</u> of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Similarly, for a smooth curve given by x=g(y), the <u>arc length</u> of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Ex. 1: Find the arc length of each graph over the given interval. Sketch and highlight the arc that you are finding the length of in the problem. (Use a calculator only when necessary).

a) $y = 2x^{3/2} + 3; [0,9]$

b) $y = \ln(\cos x); \left[0, \frac{\pi}{3}\right]$

c) $x = \frac{1}{3}\sqrt{y}(y-3); 1 \le y \le 4$

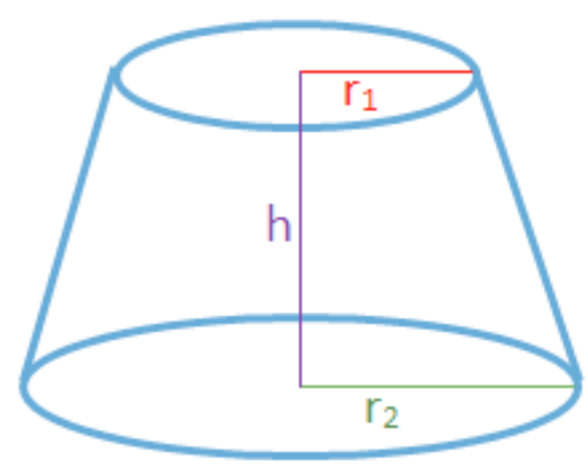
Area of a Surface of Revolution

*A surface of revolution is formed by rotating the graph of a continuous function about a line.

We use a frustum (a cross-sectional portion of the surface of a right circular cone) to set up the integral for any surface of revolution, which has a lateral surface area of

$$S = 2\pi rL$$

, where L is the length of the line segment and r is the average of the two radii.



*If we use the arc length s instead of the length of the line segment L, we can use this same formula for a curved surface of revolution.

Definition of the Area of a Surface of Revolution

Let y=f(x) have a continuous derivative [a, b]. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is $S = 2\pi \int_a^b r(x)\sqrt{1+[f'(x)]^2}\,dx$

where r(x) is the distance between the graph of f and axis of revolution. If x=g(y) on the interval [c, d], then the surface area is $S = 2\pi \int_c^d r(y)\sqrt{1+[g'(y)]^2}\,dy$

where r(y) is the distance between the graph of g and the axis of revolution.

Ex. 2: Set up an integral and evaluate the surface of revolution formed by revolving each function over the given line.

a) $y = 2\sqrt{x}; x - axis; 4 \le x \le 8$

b) $y = 9 - x^2; y - axis; 0 \le y \le 9$