Sequences (9.1)

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Sequences

A sequence $a_1, a_2, a_3, ..., a_n, ...$ is a function of n, a positive integer, and is denoted by $\{a_n\}$. The nth term of the sequence is a_n.

A recursive sequence uses the previous term in the sequence to define the new term.

Ex. 1: Write the first five terms of each sequence.

a) $a_n = -2\left(\frac{1}{2}\right)^n$

b) $a_n = \frac{n!}{2}$

c) $a_{k+1} = 3a_k + 1; a_1 = 2$ (recursive)

Limit of a Sequence

<u>Def. of the Limit of a Sequence:</u> Let L be a real number. The limit of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists M>0 such that $|a_n - L| < \varepsilon$ whenever n>M. If the limit L of a sequence exists, then the sequence *converges* to L. If the limit of a sequence does not exist, then the sequence *diverges.*

Thm. 9.1: Limit of a Sequence: Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L \quad .$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L \quad .$$

Properties of Limits of Sequences: Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty}\left(a_n \pm b_n\right) = L \pm K$

2. $\lim_{n\to\infty} ca_n = cL$, c is any real number

3. $\lim_{n\to\infty}\left(a_n b_n\right) = LK$

4. $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{L}{K}, b_n \neq 0, K \neq 0$

Ex. 2: Find the limit of each sequence with the given nth term if the sequence converges. If the sequence diverges, state so.

a) $a_n = 7 - \frac{2}{n^3}$

b) $a_n = \frac{3n^2 - n + 4}{2n^2 + 1}$

c) $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{n!}$

d) $a_n = \frac{\sin n}{n}$

Thm. 9.3: Squeeze Theorem for Sequences: If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all n>N, then

$$\lim_{n\to\infty} c_n = L \quad .$$

Thm. 9.4: Absolute Value Theorem: For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty}|a_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} a_n = 0 \quad .$$

Pattern Recognition

Ex. 3: Determine the nth term for the sequence with the first five terms given below. Then determine whether the sequence converges or diverges.

a) $-1, 2, 7, 14, 23, \ldots$

b) $\frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \frac{1}{5\cdot 6}, \frac{1}{6\cdot 7}, \ldots$

c) $1, 6, 120, 5040, 362880, \ldots$

Monotonic Sequences and Bounded Sequences

Def. of a Monotonic Sequence: A sequence $\{a_n\}$ is monotonic if its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \ldots \le a_n \le \ldots$$

or if its terms are non increasing

$$a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n \ge \ldots$$

Def. of a Bounded Sequence:

1. A sequence $\{a_n\}$ is bounded above if there is a real number M such that $a_n \le M$ for all n. The number M is called an upper bound of the sequence.

2. A sequence $\{a_n\}$ is bounded below if there is a real number N such that $N \le a_n$ for all n. The number N is called the lower bound of the sequence.

3. A sequence $\{a_n\}$ is bounded if it is bounded above and bounded below.

Thm. 9.5: Bounded Monotonic Sequences:

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Ex. 4: Use Theorem 9.5 to show that the sequence with the given nth term converges and then find its limit.

$$a_n = \frac{1}{2}\left(1 - \frac{1}{2^n}\right)$$