Taylor & Maclaurin Series (9.10)

April 2nd, 2018

Taylor Series & maclaurin Series

<u>Thm. 9.22: The Form of a Convergent Power Series:</u> If f is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all x in

an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n + ...$$

Definitions of Taylor & Mclaurin Series: If a function f has derivatives of all orders at x=c, then the series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+...+\frac{f^{(n)}(c)}{n!}(x-c)^n+...$$

is called the Taylor series for f(x) at c. Moreover, if c=0, then the series is the Maclaurin series for f.

Thm. 9.23: Convergence of Taylor Series:

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals f(x).

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

*Note: It can be proven (see section 9.1) that eventually a factorial sequence like $n!$ will grow at a faster rate than a comparable exponential sequence like x^n (with a fixed value of x, and thus

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0$$

Ex. 1: Prove that the Maclaurin series for $f(x)=\cos x$ converges to $\cos x$ for all x.

Guidelines for finding a Taylor series:

1. Differentiate f(x) several times and evaluate each derivative at c. (Look for a pattern)

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$ and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

3. Within this interval of convergence, determine whether or not the series converges to f(x) (by showing that the remainder $R_n(x)$ converges to 0).

Ex. 2: Find the Taylor series for $f(x) = \ln x$ centered at c=1. Determine its interval of convergence.

Binomial Series

Ex. 3: Find the Maclaurin series for $(1+x)^k$ and determine its radius and interval of convergence.

Deriving Taylor Series from a Basic list

*Use the basic list of power series for elementary functions on page 682 to derive each of the following series.

Ex. 4: Find the Maclaurin series for each function. (Use the table of power series for elementary functions.

A) $f(x) = \sqrt{1 + x^2}$

B) $f(x) = e^x + e^{-x}$

C) $f(x) = \cos x^{3/2}$