Power Series (9.8)

March 19th, 2018

power Series

Definition of Power Series: If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

is called a **power series**. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n + ...$$

is called a **power series centered at c**, where c is a constant.

Radius & Interval of Convergence

*If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, then the domain of *f* is the set of all x for which the power series converges

Thm. 9.20: Convergence of a Power Series: For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.

2. There exists a real number R > 0 such that the series converges absolutely for $|x-c|<R$, and diverges for $|x-c|>R$.

3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series only converges at c, the radius of convergence R=0, and if the series converges for all x, the radius of convergence is $R=\infty$. The set of all values of x for which the power series converges is the **interval of convergence** for the power series.

*To find the radius of convergence R of a power series, use the Ratio Test.

*To find the interval of convergence for a power series, you must test the endpoints individually.

Ex. 1: For each of the following series,

(i) describe the series

(ii) find the radius of convergence, or determine if the series diverges

(iii) find the interval of convergence if the series converges

(a) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{3^n}$

(b) $\sum_{n=0}^{\infty} n!x^n$

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

Differentiation & Integration of the Power Series

The. 9.21: Properties of Functions Defined by Power Series:

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + ...$$

has a radius of convergence of R>0, then, on the interval.
(c-R, c+R),
f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$

$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + ...$$

2. $\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$

$$= C + a_0(x-c) + a_1 \frac{(x-c)^2}{2} + a_2 \frac{(x-c)^3}{3} + ...$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as the original power series. The interval of convergence, however, may differ as a result of the behavior at an endpoint.

Ex. 2: Recall the convergence of the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Find the intervals of convergence for (a) *f'(x) and b)* $\int f(x)dx$