Riemann Sums & Definite Integrals (4.3)

February 13th, 2019

I. Riemann Sums

<u>Def. of a Riemann Sum:</u> Let *f be defined on the closed interval [a, b]*, and let Δ be a partition of [a, b] given by

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

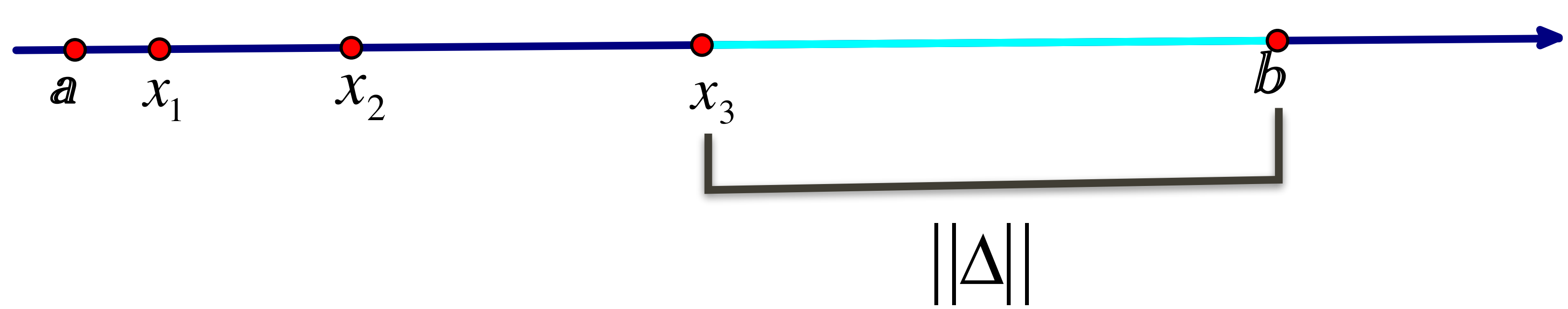
where Δx_i is the width of the *i*th subinterval. If c_i is any point in the *i*th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \qquad x_{i-1} \le c_i \le x_i$$

is called the <u>Riemann Sum</u> of f for the partition Δ .

*The norm of the partition Δ is the largest subinterval and is denoted by $\|\Delta\|$. If the partition is regular (all the subintervals are of equal width), the norm is given by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}.$$



II. Definite Integrals

Def. of a Definite Integral: If *f is defined on the closed interval [a, b]* and the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$

exists, then f is integrable on [a, b] and the limit is denoted by

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a+\frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right) = \int_a^b f(x)\,dx$$

upper limit

lower limit

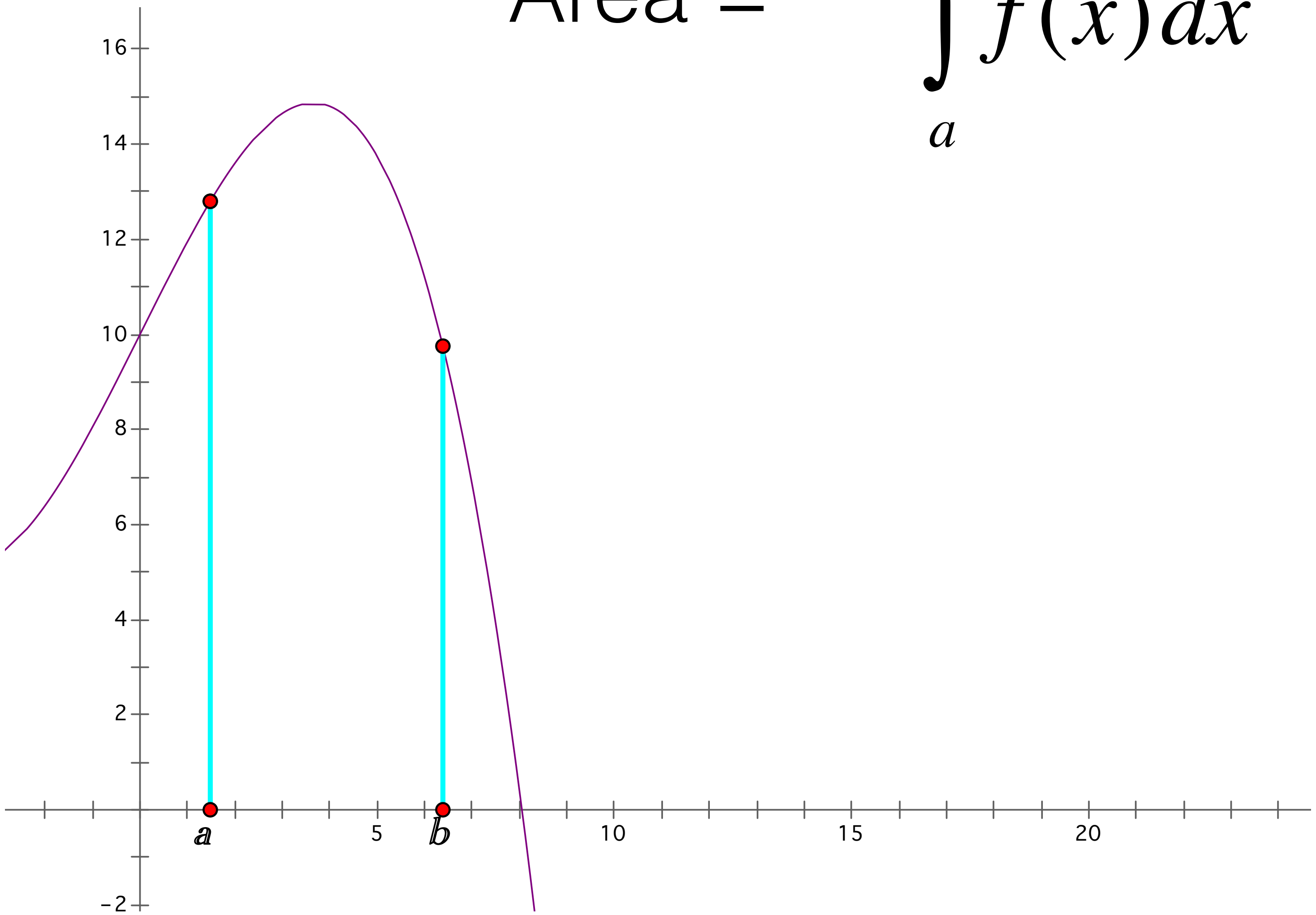
This is called the definite integral.

*An <u>indefinite integral</u> $\int f(x)dx$ is a family of functions, as seen in section 4.1. A <u>definite integral</u> $\int_a^b f(x)dx$ is a number value.

<u>Thm. 4.4: Continuity Implies Integrability:</u> If a function f is continuous on the closed interval [a, b], then f is integrable on [a, b].

Thm. 4.5: The Definite Integral as the Area of a Region: If f is continuous and *nonnegative* on the closed interval [a, b], then the area of the region bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b is given by

$$\text{Area} = \int_a^b f(x)\,dx \quad .$$



Ex. 1: Sketch a region that corresponds to each definite integral. Then evaluate the integral using a geometric formula.

a. $\int_{-2}^{2} 3\,dx$

b. $\int_{1}^{4} \frac{x}{2}\,dx$

c. $\int_{-a}^{a} (a-|x|)\,dx$

III. properties of definite integrals

Defs. of Two Special Definite Integrals:

1. If f is defined at x = a, then we define $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on [a, b], then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \quad .$$

Thm. 4.6: Additive Interval Property: If f is integrable on the three closed intervals determined by a, b, and c, where a<c<b, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thm. 4.7: Properties of Definite Integrals: If f and g are integrable on [a, b] and k is a constant, then the functions of kf and $f \pm g$ are integrable on [a, b], and

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Thm. 4.8: Preservation of Inequality:

1. If f is integrable and nonnegative on the closed interval [a, b], then $0 \leq \int_a^b f(x)\,dx$.

2. If f and g are integrable on the closed interval [a, b] and $f(x) \leq g(x)$ for every x in [a, b], then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$.

Ex. 2: Evaluate each definite integral using the following values.

$$\int_0^2 x^3\,dx = 4, \quad \int_2^4 x^3\,dx = 60, \quad \int_0^4 x\,dx = 8, \quad \int_0^4 dx = 4$$

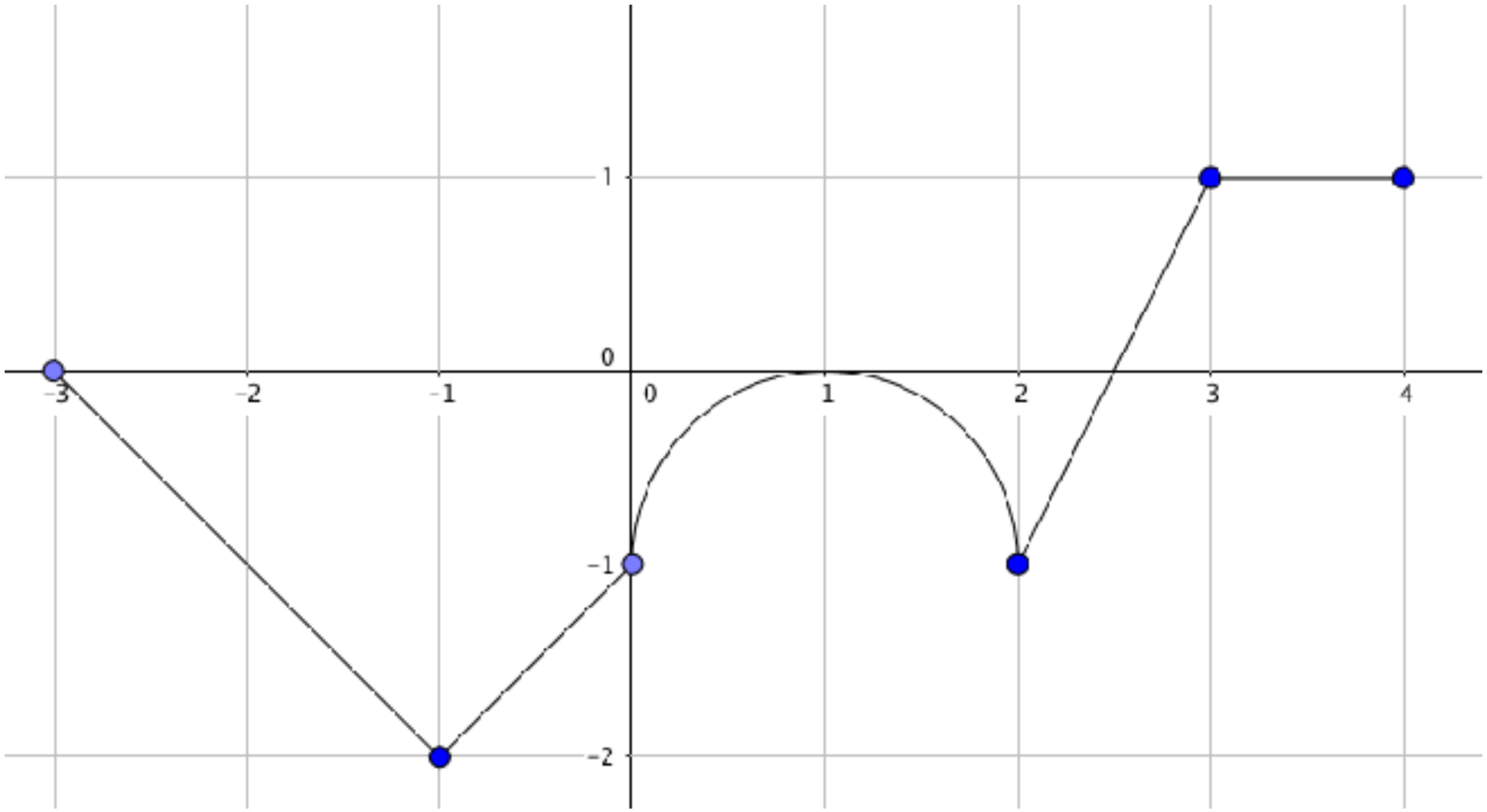
a. $\displaystyle\int_0^4 x^3\,dx$

b. $\displaystyle\int_4^4 x\,dx$

c. $\displaystyle\int_0^4 \left(x^3 - 4x + 9\right)dx$

d. $\displaystyle\int_4^0 x\,dx$

Ex. 3: Use the graph of f(x) given to evaluate the following.



$$(a)\int_{-3}^{4} f(x)\,dx$$

$$(b)\int_{4}^{2} f(x)\,dx$$

Trapezoidal Sums

Ex. 4: Use a trapezoidal sum with 4 equal subintervals to approximate the value of

$$\int_0^8 (x^2 + 1)\,dx$$